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FORMULAE FOR FREEZING OUTSIDE A CIRCULAR TUBE WITH AXIAL VARIATION OF COOLANT TEMPERATURE

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NOMENCLATURE

- Bi, Biot number, hR/k;
- c, specific heat of freezing substance;
- c_p, specific heat of coolant;
- F, area of solid in radial plane$/\pi R^2$;
- $\bar{F}$, (volume of solid between 0 and $x)/\pi R^2 x$;
- G_0, G_1, functions of F, see equations (12)–(14);
- h, convective coefficient inside tube;
- k, conductivity of freezing substance;
- $\dot{m}$, flow rate of coolant;
- NTU, number of transfer units, $2h\pi Rx/\dot{m}c_p$;
- Q, heat flow per unit length;
- $\hat{Q}$, dimensionless heat flow, $Q/2\pi Rh(T_{sat} - T_f)$;
- r^*, radius of solid–liquid interface;
- R, radius of tube;
- St, Stanton number, $h/\rho V c_p$;
- Ste, Stefan number, $c(T_{sat} - T_f)/\lambda$;
- t, time;
- T_f, coolant temperature;
- T_0, T_f at inlet;
- T_{sat}, freezing temperature;
- T_w, wall temperature of tube;
- V, velocity of coolant;
- x, axial coordinate.

Greek symbols

- ε, effectiveness;
- η, dimensionless interface radius, r^*/R;
- λ, latent heat;
- ξ, dimensionless axial coordinate, $(2\pi R/\dot{m}c_p)\int_0^x h\,dx$;
- ρ, density of freezing substance;
- ρ_f, density of coolant;
- τ, dimensionless time variable, $(k/\rho\lambda R^2)\int_0^t (T_{sat} - T_f)\,dt$.

Subscript

- 0, at inlet, $x = 0$.

INTRODUCTION

A SITUATION common to many applications is the freezing of a substance outside a coolant carrying tube. Owing to the axial increase in the temperature of the coolant as it picks up heat from the freezing substance, the frozen layer, while being axisymmetric, varies in thickness axially. Quite a few papers have addressed this problem [1–6]. Some made restrictive assumptions, and all involved numerical calculations to obtain the solution. Among these, the most thorough analysis is made in the paper by Sparrow and Hsu [6]. However, this presented no comparison with the results of the other, earlier, papers.

In this paper, we shall use the model proposed by Shamsundar and Srinivasan in ref. [3] to develop analytical solutions for the problem and evaluate the ensuing formulae using the numerical results of ref. [6].

ANALYSIS

All the models hitherto used assume that heat transfer in the freezing substance is by conduction alone. The model of ref. [3], which will be used here, differs from that of ref. [6] in neglecting heat capacities and axial conduction effects. It is the neglection of these effects that makes an analytical solution possible. The derivation of ref. [3] was with reference to a tube array and is, consequently, long and involved. We shall, therefore, give a simpler development of the model here, taking advantage of the axisymmetric nature of the particular case at hand.

The sketch of Fig. 1 will be used in the derivation. In the absence of axial conduction and with neglected heat capacities, the temperature distribution in the solid is logarithmic and the heat flow per unit length obeys the relations

$$Q = \underbrace{\frac{2\pi k(T_{sat} - T_w)}{\ln(r^*/R)}}_{\text{conduction}} = \underbrace{2\pi Rh(T_w - T_f)}_{\text{convection}}$$

$$= \underbrace{\dot{m}c\frac{\partial T_f}{\partial x}}_{\text{increase in enthalpy of coolant}} = \underbrace{\frac{\partial}{\partial t}[\pi(r^{*2} - R^2)\rho\lambda]}_{\text{latent heat released}}. \quad (1)$$

In nondimensional form, these relations reduce to

$$\hat{Q} = \frac{1}{1 + \frac{hR}{k}\ln\eta} = -\frac{\partial}{\partial\xi}\ln(T_{sat} - T_f) = \frac{1}{2Bi}\frac{\partial F}{\partial\tau}. \quad (2)$$

Equating the second and the last terms, and replacing η by $(1 + F)^{1/2}$ we get

$$2Bi\frac{\partial\tau}{\partial F} = 1 + \frac{Bi}{2}\ln(1 + F), \quad (3)$$

which shows that F is a function of τ *only*.

Next, let us calculate the axial variations of F and $(T_{sat} - T_f)$. For this purpose, consider the instantaneous energy balance for the section of the tube from 0 to x, again with heat capacities ignored

$$\underbrace{\dot{m}c_p(T_f - T_0)}_{\text{enthalpy rise}} = \underbrace{\frac{d}{dt}\int_0^x \pi\rho\lambda R^2 F\,dx}_{\text{latent heat released}}.$$

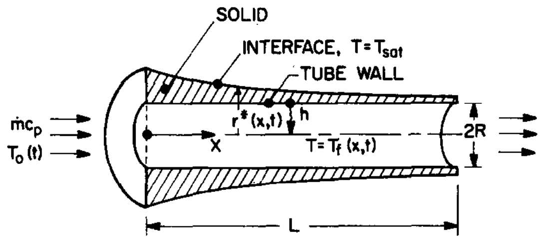

FIG. 1. Sketch showing axisymmetric solidification outside coolant-carrying tube.

Integrating with respect to time, and converting to non-dimensional variables, we get

$$\tau_0 - \tau = \frac{1}{2Bi}\int_0^{\xi} F\,\mathrm{d}\xi. \qquad (4)$$

Differentiate equation (4) with respect to ξ and combine with the last equation of the set (2) to get

$$-\mathrm{d}\ln(T_{sat} - T_f) = \frac{1}{2Bi}\frac{\mathrm{d}F}{\mathrm{d}\tau}\mathrm{d}\xi = \frac{1}{2Bi}\frac{\partial\xi}{\partial\tau}\mathrm{d}F$$
$$= -\mathrm{d}\ln F. \qquad (5)$$

Consequently, the temperature and the shape of the frozen layer vary along the axis in the same manner. That is,

$$\frac{T_{sat} - T_f}{T_{sat} - T_0} = \frac{F}{F_0}. \qquad (6)$$

Using the relation (5) in equation (2) leads us to the following relation for the axial variation of F:

$$\xi = \int_F^{F_0}\left[1 + \frac{Bi}{2}\ln(1+F)\right]\mathrm{d}\ln F. \qquad (7)$$

Finally, we may combine equations (3) and (4) to obtain the overall frozen fraction $\bar{F}$ from

$$\xi\bar{F} \equiv \int_0^{\xi} F\,\mathrm{d}\xi = 2Bi(\tau_0 - \tau)$$
$$= \int_F^{F_0}\left[1 + \frac{Bi}{2}\ln(1+F)\right]\mathrm{d}F \qquad (8)$$

and integrate equation (3) to get

$$2Bi\tau = \int_0^{F}\left[1 + \frac{Bi}{2}\ln(1+F)\right]\mathrm{d}F. \qquad (9)$$

Some features of our solution, comprising equations (6) and (7), are worth noting: (i) all the results are expressed in terms of F and F_0 as the primary variables; (ii) these equations allow the inlet temperature of the coolant to vary with time, which is important in some applications; (iii) the heat transfer coefficient h may vary axially, which is useful in some applications that involve laminar flow with its accompanying large entrance lengths. None of the earlier solutions [1–6] took both (ii) and (iii) into account. However, adaptation of the numerical scheme of ref. [6] to treat such variations is straightforward.

Several interpretations of the parameters are likely to be useful. The effectiveness of the heat exchanger is

$$\varepsilon \equiv \frac{T_f - T_0}{T_{sat} - T_0} = 1 - \frac{F}{F_0}. \qquad (10)$$

by virtue of equation (6). The variable ξ is none other than the number of transfer units (NTU), and may be expressed as follows, if h is assumed independent of x:

$$\xi = \frac{2\pi Rhx}{\dot{m}c_p} = 2\left(\frac{h}{\rho_f V c_p}\right)(x/R) = 2St(x/R) \qquad (11)$$

where St is the Stanton number. Sparrow and Hsu [6] fixed $(x/R) = 100$, and presented results for $St = 0.003$ and 0.005. Thus, our neglecting heat capacity and axial conduction leads to a solution in which St and (x/R) combine into a single parameter.

For convenience in obtaining numerical results, let us introduce the auxiliary functions

$$G_0(F) = \int_0^F \ln(1+F)\,\mathrm{d}F = (1+F)\ln(1+F) - F, \qquad (12)$$

$$G_1(F) = \int_0^F \ln(1+F)\frac{\mathrm{d}F}{F}$$
$$= \frac{1}{2}[\ln(1+F)]^2 + \int_0^F \frac{\ln(1+F)\,\mathrm{d}F}{(1+F)F}. \qquad (13)$$

The last integral on the RHS of equation (13) is the Debye function of order one* with $\ln(1+F)$ as the argument, and is tabulated in, for example, ref. [7]. Alternatively, the following Padé approximation may be used with a programmable hand calculator:

$$G_1(F) = [p(1 + 17p^2/450)/(1 + p^2/100) + p^2/4](1 \pm \delta)$$

where

$$p = \ln(1+F),$$
$$\delta < 10^{-5},\ F < 3,$$
$$\delta < 10^{-4},\ F < 8 \qquad (14)$$

and

$$\delta < 10^{-3},\ F < 29.$$

In terms of these functions, our final results are

$$\tau = F/2Bi + G_0(F)/4, \qquad (15)$$

$$\xi = NTU = 2\,St(x/R)$$
$$= \ln(F_0/F) + (Bi/2)[G_1(F_0) - G_1(F)], \qquad (16)$$

$$\varepsilon = (T_f - T_0)/(T_{sat} - T_0) = 1 - F/F_0 \qquad (17)$$

$$\xi\bar{F} = (F_0 - F) + (Bi/2)[G_0(F_0) - G_0(F)]$$
$$= 2Bi(\tau_0 - \tau). \qquad (18)$$

COMPARISON OF RESULTS

As noted earlier, our results are expressed using the fraction F as the primary variable. In practice, however, one wishes to obtain the results in terms of t and x. To do so, the following steps should be undertaken:

(i) From the known variation (if any) of T_0 with t, calculate τ_0 using the definition.

(ii) Use this value of τ_0 in equation (15), written at $x = 0$, and solve the equation for F_0.

(iii) Calculate $\xi = NTU$ from the data, and solve equation (16) for F.

(iv) Calculate ε and the fluid temperature T_f from equation (17). If desired, calculate T_w from equation (1).

(v) Calculate $\bar{F}$ from equation (18), if desired.

Of these steps, (ii) and (iii) involve the solution of nonlinear equations which, since the equations are analytical, is best done using the Newton–Raphson method.

* The Debye theory of specific heat of solids leads to the Debye integral of order three, which has $\ln(1+F)$ raised to the third power instead of the first as here.

If complete design charts are needed, it is easier to assume various values of ε and F_0, and to directly calculate NTU, τ_0 and $\bar{F}$.

Many useful conclusions can be drawn by merely inspecting the analytical results. For instance, if the Biot number is small, $\xi \approx \ln(F_0/F) = -\ln(1-\varepsilon)$ or $\varepsilon \approx 1 - e^{-NTU}$, which is the effectiveness of a two-fluid heat exchanger with one fluid changing phase with very high film coefficient. The larger the Biot number is, the more significant is the deterioration in effectiveness as solidification progresses. Since detailed results have already been given by Sparrow and Hsu [6] from which the various influences are readily obtainable, we shall only make a comparison, referring the reader to ref. [6] for figures and a discussion of the results.

Using the procedure stated above, we calculated tables of results for all* the cases dealt with by ref. [6] (Bi = 2,5, Ste = 0, St = 0.003 and 0.005, x/R = 0–100). By plotting these numbers on full-page copies of the original figures of ref. [6] on graph paper, we found the predictions of our equations to coincide with the results of ref. [6], to an accuracy of better than 0.2% (plotting accuracy). This agreement shows that the neglection of axial conduction is proper and that the model of ref. [3], including the remarkable result, equation (6), is correct. On the other hand, the agreement also attests to the care taken in the numerical work of ref. [6] and the accuracy of the results presented there. Finally, there is no need to present any figures of results, since those of the results of ref. [6] that pertain to $Ste \approx 0$ exactly do this.

A feature of the problem, which is elucidated by the formulae, is worth mentioning. For large Biot numbers, ref. [6] shows interfaces that are straight in the axial direction. That this is the case is assumed in the analysis of ref. [5].

From equation (7), we obtain

$$-\frac{\partial\eta}{\partial\xi} = \frac{\eta^2-1}{2\eta(1+Bi\ln\eta)} \approx \frac{\eta^2-1}{2Bi\,\eta\ln\eta}$$

for large Bi. The expression $(\eta^2-1)/(\eta\ln\eta)$ varies very little with η (from 2 at $\eta = 1$ to 2.16 at $\eta = 2$). Consequently, the slope becomes constant along the axis at any instant. Since $F = \eta^2 - 1$, equation (6) shows that the temperature distribution is parabolic in x.

A small-time solution based on a linear temperature distribution in the radial direction was presented and compared in ref. [6]. Our solution leads to slightly more complicated formulae, but is not restricted to small time, and is in excellent agreement with the finite-difference results.

* With the exception of a few results for Ste = 1, shown in Figs. 4 and 11 of ref. [6]. Our model sets $Ste \approx 0$, and cannot predict any effect of Ste; however, even for the large value Ste = 1, ref. [6] shows a change from the Ste = 0 predictions of less than 11% in the thickness of the frozen layer and less than 9% in the energy extracted.

CONCLUSIONS

Approximate closed-form formulae have been derived for freezing of a substance outside a coolant carrying tube, by neglecting heat capacity and axial conduction. The coolant inlet temperature may vary with time and the convective coefficient may vary axially, but the latter, as well as the flow of the coolant, are assumed to be constant in time. The temperature distribution is logarithmic in the radial plane. The axial temperature distribution is identical with the distribution of F, the solid fraction in the radial plane, along the axis. For large Biot number, the interface is a truncated cone, and the axial temperature distribution is parabolic.

The formulae are in excellent agreement with the results of finite-difference calculations and detailed results are available [6]. As the finite-difference calculations included heat capacity and axial conduction, the agreement justifies neglecting those effects.

Acknowledgement—Professor Sparrow was kind enough to provide the original figures of ref. [6], for which I am grateful.

REFERENCES

1. D. J. Morrison and S. I. Abdel-Khalik, Effects of phase-change energy storage on the performance of air-based and liquid-based solar heating systems, *Sol. Energy* **20**, 57–67 (1978).
2. T. F. Green and G. C. Vliet, Thermal response of a latent heat storage unit: An analytical and experimental investigation, ASME paper 79-HT-36. A.S.M.E., New York (1979).
3. N. Shamsundar and R. Srinivasan, Effectiveness-NTU charts for heat recovery from latent heat storage units, *J. Sol. Energy Engng* **102**, 263–271 (1980).
4. M. B. Lambdon and S. I. Guceri, Heat transfer of phase change materials in two-dimensional cylindrical coordinates, AIAA Paper 81-1046. A.I.A.A., New York (1981).
5. G. Grossman and D. Pesotchinski, A two-dimensional model for thermal energy storage in a phase-changing material interacting with a heat carrying fluid, ASME paper 81-WA/HT-35, A.S.M.E. (1981).
6. E. M. Sparrow and C. F. Hsu, Analysis of two-dimensional freezing on a coolant-carrying tube, *Int. J. Heat Mass Transfer* **24**, 1345–1357 (1981).
7. I. A. Stegun, Miscellaneous functions, in *Handbook of Mathematical Functions* (edited by M. Abramowitz and I. A. Stegun). U.S. National Bureau of Standards (1964).